RESEARCH STATEMENT

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My research subjects lie under the illusive name - *mathematical physics*. My main subject of research is *spectral geometry of metric graphs*, also known as *quantum graphs*. Much of my work is motivated from *quantum chaos*. My research includes questions and methods that relate to other mathematical disciplines such as

(1) Graph theory.
(2) Statistics and Probability in high dimensions.
(3) Riemannian Geometry.
(4) Stable Polynomials (real algebraic geometry).
(5) Dynamics and Ergodic theory.

The research statement is divided into two sections, the first regards metric graphs and is partitioned according to my recent works: *universal nodal count* [ABB21] joint with Band and Berkolaiko, *Neumann domains* [AB21] joint with Band, and a work in progress on *genericity* [Alo]. The second section describes a new work in progress, joint with Cynthia Vinzant, on *Quasi-Crystals* using a recent construction of Kurasov and Sarnak [KS20] in terms of *stable polynomials*. These technical terms and the Kurasov-Sarnak construction are explained and motivated in the second section.

Before discussing metric graphs, let me introduce two open problems in spectral geometry and quantum chaos, that serve as a motivation for my work. Consider the "simple" case of domains in $\mathbb{R}^2$, i.e. *planar domains*. The spectral properties are commonly given in terms of eigenvalues and eigenfunctions of the Laplacian with Dirichlet boundary conditions. An example of relation between the geometry and the spectrum of a planar domain Ω is the *Weyl law* [Wey11], by which the asymptotic growth rate of the spectrum is $\frac{1}{4\pi}$ area(Ω). A more settle question concern the fluctuations around the asymptotic growth, which can be captured by the gaps between consecutive eigenvalues. A famous open problem, formally stated by Bohigas-Giannoni-Schmit [BGS84], says (roughly)

Conjecture 0.1. [BGS84] *If the "billiard flow" on a planar domain is* ***chaotic****, then the (properly normalized) gaps distribution is* ***universal****, and agree with that of GOE random matrices.*

This was the starting point of the field of *quantum chaos*, as described by Berry in [Ber87]. In fact, in [BGS84] a broader class of chaotic systems was considered, but later on counter examples were found, see [Sar93]. Berry also conjectured that the eigenfunctions of a chaotic system should have universal behaviuor. Their local behaviuor should agree with random (Gaussian) linear combinations of plain waves (often called *random waves*). An example of eigenfunction's property that is believed to fluctuate in a universal manner is the *nodal count*. Given a planar domain Ω and its n-th eigenfunction f_n, the nodal count ν_n is the number of connected components of $\{x \in \Omega : f_n(x) \neq 0\}$. Blum-Gnutzmann-Smilansky [BGS02] and Bogomolni-Schmit [BS02] conjectured:

Conjecture 0.2. [BGS02, BS02] *Given a* ***chaotic*** *domain in the plane, its nodal count ν_n is distributed as a Gaussian with mean and variance of order n.*

This conjecture is still open, but progress was made by Nazarov-Sodin regarding the nodal count of random waves, see [NS09, NS20].

The name "*quantum graph*" first appeared in the work of Kottos and Smilansky [KS97] where they considered a model of a 1D Laplacian on a metric graph with $\mathbb{Q}$-independent[1] edge lengths (we will refer to those as metric graphs). Kottos and Smilansky suggested that such metric graphs are good paradigm for quantum chaos, as numerically their spectrum has GOE like behaviuor, as expected from chaotic systems by Conjecture 0.1. Barra and Gaspard [BG00], gave a description of the spectrum of such system using an ergodic map on a hypersurface in a high-dimensional torus. Using which they found a deviation of the gaps distribution from GOE. However, they conjectured that this deviation from GOE should go to zero as the graph structure grows. Hence, a more appropriate paradigm for chaos was suggested - **growing sequence of metric graphs with $\mathbb{Q}$-independent lengths**.

1. Spectral geometry of metric graphs (quantum graphs)

A *metric graph* is a 1D manifold with singularities. The singular points are the vertices of the graph and every edge is a 1D segment. We denote such a metric graph by (Γ, ℓ), with graph structure given by Γ, a finite connected (discrete) graph[2] of $E > 1$ edges. The metric is determined by the vector of edge lengths $\ell = \{l_e\}_{e=1}^E \in \mathbb{R}_+^E$. On the metric graph we define a 1D Laplace operator with Neumann-Kirchhoff vertex conditions. The spectrum of (Γ, ℓ) is then the sequence of (square-root) eigenvalues denoted by

$$0 = k_0 < k_1 \leq k_2 \leq k_3 \ldots \nearrow \infty$$

including multiplicity. This model, and its generalization to any Schrödinger operator acting along the edges, appeared in various scientific disciplines in the last few decades, modeling complex phenomena such as superconductivity in granular and artificial materials [Ale83], acoustic and electromagnetic wave-guide networks [BK03] and Anderson localization [CMV06, SS00] to name but a few. We will only consider the the "simple" geometric setting of Laplacian (without magnetic or electric potential) and Neumann-Kirchhoff vertex conditions. This model which we will call a *metric graph* already serves as non-trivial one-dimensional model for spectral geometry, where exotic mathematical phenomena can often occur. For example, some graph structures have frequent appearance of scars (unusual localization of eigenfunctions) [BKW04, CdV15]. Another example, number theoretic in nature, is a recent work by Kurasov and Sarnak on the arithmetic structure of the spectrum of a metric graph. They show that if the edge lengths are $\mathbb{Q}$-independent, then the spectrum is infinite dimensional over $\mathbb{Q}$ and has a bound on the maximal length of arithmetic progressions in it.

1.1. Universal nodal count for metric graphs.

Conjecture 0.2 says that for a chaotic domain, the nodal count ν_n has a **universal** behaviuor:

$$\nu_n = Cn + c\sqrt{n}\sigma,$$

[1] The term $\mathbb{Q}$-independent means that there is no (non-trivial) solution $q \in \mathbb{Q}^E$ to $\ell \cdot q = 0$.

[2] For ease of presentation we restrict the discussion to simple graphs, i.e. every edge has a unique pair of two distinct vertices. We also assume no vertices of degree 2 as these are removable singularities.

where the constants C, c may depend on the domain and $\sigma \sim N(0,1)$ has normal distribution. Namely, for any $a < b$

$$\lim_{N\to\infty} \frac{\left|\left\{n \le N \,:\, a \le \frac{\nu_n - Cn}{c\sqrt{n}} \le b\right\}\right|}{N} = P(a \le \sigma \le b)$$

However, we do not even know how to prove that the above limits exist.

In the case of metric graphs, we conjecture that a similar universal behavior accrues in the limit of large graphs. Unlike the 2D case, for graphs we can show that the analog limits exist and even prove the Gaussian behaviuor for certain families of graphs [ABB18, ABB21]. The nodal count ν_n for a metric graph (Γ, ℓ) is the number of connected components of $\{x \in (\Gamma, \ell) \,:\, f_n(x) \neq 0\}$. The nodal count for metric graphs exhibit an upper bound similar to that of the 2D case, but also a lower bound, unlike the 2D case.

$$n - \beta \le \nu_n \le n,$$

where $\beta = \text{rank}(\pi_1(\Gamma))$ is the first Betti number of the graph (intuitively, the number of cycles). The upper bounded was proved in [GSW04] (applying Courant's method [Cou23] to the metric graph), and the lower bound was proven in [Ber08]. Motivated by Conjecture 0.2, we define a distribution σ over $\{0, 1, \dots, \beta\}$ such that ν_n is distributed as

$$\nu_n = n - \sigma,$$

in the following sense:

$$P(\sigma = j) := \lim_{N\to\infty} \frac{|\{n \le N \,:\, \nu_n = n - j\}|}{N}.$$

The existence of these limits is obtained in [ABB18], assuming $\mathbb{Q}$-independent lengths, which we will always assume. The proof relies on the "nodal-magnetic" relation [BW14] which characterize the deviation $n - \nu_n$ as a Morse index of the eigenvalue k_n under magnetic perturbations. Then, applying an ergodicity argument from [BG00] to replace the average over the spectrum with integration over a certain hypersurface in an E-dimensional torus. We also show in [ABB18] that σ is symmetric with mean $\mathbb{E}(\sigma) = \frac{\beta}{2}$. We call σ the *nodal surplus distribution* and denote it by $\sigma(\Gamma, \ell)$ to emphasize its (Γ, ℓ) dependence. By identifying a certain family of symmetries of this construction, we obtained $\sigma(\Gamma, \ell)$ for a specific family of growing graphs with $\mathbb{Q}$-independent ℓ's.

Theorem 1.1. [ABB18] *If Γ has vertex-disjoint cycles and ℓ is $\mathbb{Q}$-independent, then $\sigma(\Gamma, \ell)$ is Binomial with parameters $Bin(\beta, \frac{1}{2})$. In particular, it converges to a Gaussian as $\beta \to \infty$.*

In a recent work [ABB21], we formulate Conjecture 0.2 for metric graphs.

Conjecture 1.2. [ABB21] *The distance between $\sigma(\Gamma, \ell)$ and the Gaussian distribution of same mean and variance goes to zero as $\beta \to \infty$, uniformly over all (Γ, ℓ) with Betti number β and $\mathbb{Q}$-independent ℓ.*

We also conjecture that the variance growth is of order β. A detailed and more quantified statement is given in [ABB21]. In [ABB21] we prove the conjecture for two more families of graphs. We also show that $\sigma(\Gamma, \ell)$ is convex in ℓ. We provide an upper bound, $C(\Gamma)$, to the distance of $\sigma(\Gamma, \ell)$ from the relevant Gaussian. This bound is uniform in ℓ, and can be numerically evaluated (efficiently). In [ABB21] we calculated $C(\Gamma)$ numerically for

25 different graph structures, including random and deterministic, and a clear decrease of $C(\Gamma)$ in terms of β is evident.

1.1.1. *Future work and questions.*

(1) Are there any relations between the nodal surplus distributions of a graph and its sub-graphs?
(2) Are there any models of random graphs for which $\sigma(\Gamma, \ell)$ can be estimated?
(3) In the proof of Theorem 1.1, the symmetries are related to a "cut of one edge" (an edge whose removal disconnects the graph), and result in a partition of σ to a sum of uncorrelated random variables. This can be extended to graphs with disjoint clusters of cycles using martingale CLT results.
(4) We also believe that the method of 1-edge cuts can be extended to larger cuts, which will result in a partition of σ into a sum of weakly correlated random variables. This may lead to a proof of the Gaussian limit for clusters of cycles.

1.2. **Neumann count.** The concept of a *Neumann partition* was introduced independently in [Zel13, MF14], in analogy to nodal partitions of manifolds, and was further developed in [BF16, BET17]. The Neumann partition is a Morse partition of the manifold according to an eigenfunction. Connected components of the Neumann partition are called Neumann domains. The name "Neumann domain/partition" reflects the fact that a restriction of an eigenfunctions to its Neumann domain is an eigenfunction of that Neumann domain with Neumann boundary conditions. We define a metric graph analog in [ABBE20, AB21]. The n-th Neumann partition of a metric graph (Γ, ℓ) is given by removing the critical points along the edges. We denote the Neumann count by μ_n, namely the number of connected components of $\{x \in (\Gamma, \ell) \,:\, f_n'(x) \neq 0\}$. In [AB21], we provide topological upper and lower bounds on the Neumann count:

$$n + 2 - 2\beta - |\partial\Gamma| \leq \mu_n \leq n + \beta,$$

where $|\partial\Gamma|$ is the number of degree 1 vertices of Γ. Similarly to the nodal count, we adopt a probabilistic setting:

$$\mu_n = n - \omega,$$

where ω is a distribution on $\{-\beta, \dots, 2\beta + |\partial\Gamma| - 2\}$ such that

$$P(\omega = j) := \lim_{N\to\infty} \frac{|\{n \leq N \,:\, \omega(n) = j\}|}{N}.$$

We show in [AB21] that these limits exist for any graph with $\mathbb{Q}$-independent ℓ, and that ω is symmetric around its mean $\mathbb{E}(\omega) = \frac{\beta + |\partial\Gamma| - 2}{2}$. This result has implications to **inverse problems**:

(1) The nodal count and Neumann count provide different information on the graph structure. For example, $\sigma \equiv 0$ for every tree graph (since $\beta = 0$), however ω can distinguish between trees of different $|\partial\Gamma|$, since $\mathbb{E}(\omega) = \frac{|\partial\Gamma| - 2}{2}$.
(2) Given access to both $\mathbb{E}(\omega)$ and $\mathbb{E}(\sigma)$, we get β and $|\partial\Gamma|$ and thus bounding the size (number of edges and vertices) of the graph structure.

Similarly to the nodal distribution, we conjecture that the Neumann distribution has a universal Gaussian behaviuor as $\beta + |\partial\Gamma|$ grows to infinity. Following [ABB18] we prove a binomial result. Call a graph $(d, 1)$-regular if its vertex degrees are either d or 1.

Theorem 1.3. [AB21] *If Γ is a $(3,1)$-regular finite tree and ℓ is $\mathbb{Q}$-independent, then $\omega(\Gamma,\ell)$ is Binomial with parameters $Bin(|\partial\Gamma|-2,\frac{1}{2})$. In particular, it converges to Gaussian with $|\partial\Gamma|\to\infty$.*

1.2.1. *Future work and questions.*

(1) Are there two different graphs with the same nodal and Neumann count?
(2) Is there a Neumann count analog to the nodal-magnetic theorem [BW14]. The method of proof of 1.3 suggests that the analog of magnetic flux should relate to degree 3 vertices.
(3) Numerically, it seems that the bounds $-\beta\le\omega_n\le 2\beta+|\partial\Gamma|-2$ are not optimal. We conjecture in [AB21] that better bounds hold $0\le\omega_n\le\beta+|\partial\Gamma|-2$.

1.3. **Generic properties of eigenvalues and eigenfunctions.** Fixing a graph structure Γ and letting the metric ℓ change, we can consider the pairs (k_n,f_n) as functions of ℓ. When considering properties of eigenfunctions, we neglect the trivial pair $k_1=0$ with constant eigenfunction $f_1\equiv C$. As a motivational example, consider the seminal genericity result of Uhlenbeck on compact Riemannian manifolds [Uhl72], which says that given a fixed manifold, for a Baire-generic choice of smooth metric, every pair of eigenvalue λ_n with eigenfunction f_n satisfies:

(1) λ_n is a simple eigenvalue (no multiplicity).
(2) f_n is Morse. That is, $\nabla f_n(x)=0 \Rightarrow$ the Hessian of f_n at x is invertible.
(3) 0 is not a critical value of f_n. Namely, $f_n(x)=0 \;\Rightarrow\; \nabla f_n(x)\neq 0$.

Similarly, Friedlander [Fri05] and Berkolaiko-Liu [BL17] proved that given a graph structure Γ, for a Baire-generic choice of ℓ, every (k_n,f_n) pair of (Γ,ℓ) satisfies properties[3] (1), (2) and (3) as above, and in addition

(4) f_n does not vanish on vertices.

In [ABB18] we show that the implicit set of "good" ℓ's, can be replaced by an explicit criteria, if we relax *all pairs* to *almost every pair*, in the sense of a density 1 subsequence.

Theorem 1.4. [ABB18] *Given a graph Γ, for any $\mathbb{Q}$-independent ℓ, almost every pair of (Γ,ℓ) satisfies properties (1),(2),(3) and (4).*

Let $G\subset\mathbb{R}_+^E$ be the set of ℓ's satisfying all of the properties above. Then, Baire-genericity means that G contains a countable intersection of open-dense sets. A priori such a set may have zero Lebesgue measure. Call G *strongly-generic* if its complement $\mathbb{R}_+^E\setminus G$ is contained in a countable union of lower dimensional varieties. A strongly-generic G is Baire-generic and has **full Lebesgue measure**.

Theorem 1.5. [Alo] *Given a graph Γ, the set of ℓ's for which properties (1),(2),(3) and (4) hold for all pairs is **strongly-generic**.*

One way of interpreting (4) is saying that generically eigenfunctions do not satisfy the "extra vertex condition" $f_n(v)=0$. Vertex conditions are usually linear relations imposed on $\mathrm{trace}(f_n)$, the vector of all values and normal derivatives of f_n at the vertices. Let us generalize the vertex conditions to a broader notion of homogeneous[4] relations: $q(\mathrm{trace}(f_n))=0$ where q is a homogeneous polynomial. We say that a relation q is trivial (on Γ) if $q(\mathrm{trace}(f_n))=0$ for all pairs of (Γ,ℓ), for any ℓ. In [Alo] we prove that

[3]To be precise, (1) is proved in [Fri05], and (4) in [BL17]. Properties (2) and (3) are not stated but follow trivially from (4).

[4]Homogeneous because we care about eigenfunctions up to a multiplicative constant

Theorem 1.6. [Alo] *Given a graph* Γ *and non-trivial homogeneous relation* q*,*

(a) the set of ℓ*'s for which* $q \neq 0$ *on all* (Γ, ℓ) *pairs is* ***strongly-generic****.*
(b) for any $\mathbb{Q}$*-independent* ℓ*,* $q \neq 0$ *on almost any pair of* (Γ, ℓ)*.*

Future work:

(1) Working on a conjecture of Sarnak, saying that generically the spectrum of (Γ, ℓ) should be $\mathbb{Q}$-independent.
(2) Working on a conjecture of Quantum Unique Ergodicity for graphs with $\mathbb{Q}$-independent ℓ when the graph size and complexity grows to infinity.

2. Quasi-crystals and Stable Polynomials

A crystal, from a physical point of view, is a system of atoms in a lattice structure, such as metals and semi-conductors for example. The lattice structure can be experimentally observed by a scattering experiment, the diffraction pattern would have peaks at the dual lattice locations. Mathematically, this phenomena is described by the following duality, known as *Poisson summation formula*. If Λ is a lattice whose dual lattice is S, then the distribution μ defined as a sum of unit mass atoms along the points of Λ, has a Fourier transform $\widehat{\mu}$ which is a distribution with unit mass atoms along the points of S. In other words, for any rapidly decaying f with Fourier transform $\hat{f}$,

$$\sum_{x\in\Lambda} f(x) = \sum_{k\in S} \hat{f}(k).$$

One may ask are there any other pairs of such "dual" discrete sets Λ and S for which such a relation holds? Generalization of this form were considered by Meyer in the 70's but it was believed that no such physical system exists. A decade later, Shechtman discovered a *Quasi-crystal* in an experiment[5], a discovery for which he was awarded the 2011 Nobel prize. In the context of metric graphs, a summation formula of similar nature was introduced in dependently in [KS97] and [Rot95]. Given a metric graph (Γ, ℓ) with (square-root) eigenvalues k_n for $n \in \mathbb{N}$, call its set of periodic orbits $\mathcal{P}(\Gamma)$ and for each periodic orbit γ let ℓ_γ be its length. Then for any "nice enough" function f with Fourier transform $\hat{f}$,

$$\sum_{n=1}^{\infty} f(k_n) = \sum_{\gamma\in\mathcal{P}(\Gamma)} c_\gamma \hat{f}(\ell_\gamma), \tag{2.1}$$

where c_γ are some fixed complex coefficients. Kurasov and Sarnak had showed that (2.1) holds for all rapidly decaying function f and that the coefficients grow slowly $\sum_{\gamma\in\mathcal{P}} |c_\gamma| \hat{f}(\ell_\gamma) < \infty$. The associated measure $\mu = \sum_{n=1}^{\infty} \delta_{k_n}$ in such case is said to be a *Fourier Quasi-Crystal*, see [LO15]. This led Kurasov and Sarnak to the following construction of such quasi-crystals. We call $p \in \mathbb{C}[z_1, z_2 \ldots z_n]$ a *stable polynomial* if $p(z_1, z_2 \ldots z_n) \neq 0$ whenever all $|z_j| < 1$ or all $|z_j| > 1$. Consider the notation $\exp(i\vec{x}) := (e^{ix_1}, \ldots, e^{ix_n})$, and for any $\ell \in \mathbb{R}^n_+$ let $\mu_{p,\ell}$ be the measure given by a sum of unit mass atoms on the zero set $\{k \in \mathbb{R} : p(\exp(ik\ell)) = 0\}$. Then $\mu_{p,\ell}$ is a Fourier quasi-crystal [KS20]. Since p is stable, then the trigonometric polynomial $F(k) := p(\exp(ik\ell))$ has only real roots, in which case it is called a *real rooted* trigonometric polynomial. Following [KS20], Olevskii and Ulanovskii showed in [OU20] that a measure μ which is a

[5] A scattering experiment where the diffraction pattern had sharp peaks with a 5-fold symmetry which no lattice can have.

sum of unit mass atoms on a discrete set Λ is Fourier quasi-crystal if and only if Λ is the zero set of a real rooted trigonometric polynomial. A priori, the construction of Olevskii and Ulanovskii may contain measures that are not coming from stable polynomials. However,in a work in progress, joint with Cynthia Vinzant, we conjecture that this is not the case.

Conjecture 2.1. [AV] *Given any real rooted trigonometric polynomial $F(k)$,*

*(1) There exists a **stable** polynomial $p \in \mathbb{C}[z_1, z_2 \dots z_n]$ and a **positive** vector $\ell \in \mathbb{R}^n_+$ such that $F(k) = p(\exp(ik\ell))$.*
(2) If $F(k)$ is not periodic, then p and ℓ can be chosen such that ℓ is $\mathbb{Q}$-independent.

If true, it would mean that the space of all such Fourier quasi-crystals can be constructed out of the space of stable polynomials which has been studied. In [AV] we provide a "dictionary" between properties of stable polynomials and properties of Fourier quasi-crystals. We also show that some spectral properties of metric graphs can be translated to the Fourier quasi-crystals context. As an example, the Weyl law has the following analog:

Theorem 2.2. [AV] *Let p be a stable polynomial, ℓ a positive vector and $\mu_{p,\ell}$ the corresponding constructed measure. Let $\boldsymbol{d} = (d_1, d_2 \dots, d_n)$ be the degrees of p in each variable. Then for any $x \in \mathbb{R}$ and $R > 0$,*

$$\mu_{p,\ell}([x, x+R]) = \frac{\boldsymbol{d} \cdot \ell}{\pi} R + \text{ error term},$$

with a uniform bound $|\boldsymbol{d}| = \sum d_j$ on the error term.

Another example is a theorem recently proved by Kurasov and Sarnak on the spectrum of metric graphs that translates to this context.

Theorem 2.3. [AV] *Let $p, \ell, \Lambda_{p,\ell}$ and $\boldsymbol{d}$ as in the previous theorem. Further assume that p is irreducible and has more then 2 monomials. Then $\Lambda_{p,\ell}$ has infinite dimension over $\mathbb{Q}$ and does not contain any arithmetic progression of length bigger then $C(\boldsymbol{d})$.*

The constant $C(\mathbf{d})$ is given explicitly and depends only on $\mathbf{d}$. More examples include

(1) We prove the existence of a gap distribution in analogy to the work of Barra and Gaspard [BG00].
(2) We prove a relation between the infimum of the gaps and the singularities of the zero variety of p in $\mathbb{C}^n$. In particular, if p is non singular, then $\Lambda_{p,\ell}$ is uniformly discrete[6].

2.1. Future work.

(1) Working on Conjecture 2.1 using tropical geometry.
(2) By analyzing the gap distribution of $\mu_{p,\ell}$ as a function of p and ℓ, we may be able to achieve progress in Conjecture 0.1 for metric graphs.
(3) Constructing a model of random Fourier Quasi-Crystals in terms of random stable polynomials. An example can be $p(\vec{z}) = \det(1 - \operatorname{diag}(\vec{z})U)$ where U is a random unitary matrix, in which case we believe that the gap distribution will behaves like CUE.
(4) Higher dimensional analogs. Is it possible to create quasi-crystals in higher dimensions using stable varieties of higher co-dimension?

[6] In fact a *Delone* set.

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