RESEARCH SUMMARY

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My main subject of research is the relation between geometry and spectral properties of quantum graphs. A quantum graph is a metric graph equipped with a Schrödinger operator and a choice of vertex conditions such that the operator is self-adjoint. In the last few decades quantum graphs appeared in various scientific disciplines, modeling complex phenomena such as superconductivity in granular and artificial materials [1], Anderson localization [13], acoustic and electromagnetic waveguide networks [8], and quantum chaos [12, 17]. From the mathematical point of view, quantum graphs serve as non-trivial one-dimensional models for spectral geometry. My main work can be categorized into the following:

Nodal statistics and its universal Gaussian limit conjecture. The study of nodal sets of Laplace eigenfunctions is an integral part of spectral geometry. Nodal domains are the connected components of the nodal set's complement, namely the regions on which the function has a fixed sign. One of the earliest results in spectral geometry given by Courant in [15] showed that the number of nodal domains of the n^{th} eigenfunction is bounded by n. The number of nodal domains is commonly known as the *nodal count*. Courant's bound on the nodal count led to many questions regarding the asymptotic behavior of the nodal count. It was also asked whether the Courant bound is sharp and if so how many eigenfunctions achieves the upper bound (these eigenfunctions are known as Courant sharp). In the case of a quantum graph, it is more convenient to count the number of nodal points $\phi_n := \#\{x : f_n(x) = 0\}$ instead of the number of nodal domains (for high enough eigenvalue the two quantities are the same up to a global constant). For the latter both an upper bound [16] and a lower bound [9] were obtained and can be written as

$$n \le \phi_n \le n + \beta,$$

where β is the first Betti number of the graph. We call the deviation of ϕ_n from n the *nodal surplus* $\sigma_n := \phi_n - n$. Given the global constant bounds on the nodal surplus the asymptotic growth of the nodal count is $\phi_n \sim n$, and its fluctuations around the linear growth, namely the nodal surplus, can be statistically investigated. Band showed in [4] that $\sigma_n \equiv 0$ if and only if the graph is a tree, namely trees are the only graphs for which all eigenfunctions are Courant sharp. In [2] we showed that for every possible value $0 \le j \le \beta$ the following limit exists

$$P(\sigma_n = j) = \lim_{N\to\infty} \frac{\#\{n \le N : \sigma_n = j\}}{N}$$

and that under proper normalization it defines a nodal surplus probability distribution. We also showed that these distributions are symmetric $P(\sigma_n = j) = P(\sigma_n = \beta - j)$ and as a corollary one can find the first Betti number of a graph from its average nodal surplus which generalizes Band's result.

Although we have no tools for computing the nodal surplus distribution for a general graph, we showed in [2] that for a particular family of graphs which we call *edge separated* the nodal surplus distribution is binomial $Bin\left(\beta, \frac{1}{2}\right)$. This result implies that for any sequence of edge separated graphs with $\beta \to \infty$ the properly normalized nodal surplus distribution will converge to a normal distribution by the central limit theorem. An intense numerical study led us to the following.

Conjecture. *For every sequence of graphs with increasing first Betti number, the sequence of properly normalized nodal surplus distributions will converge to a normal distribution.*

In a recent work in progress we have been able to prove the conjecture for several families of graphs but the conjecture is still open.

The nodal-magnetic connection and the dispersion relation manifold. The fascinating relation between the nodal count and magnetic stability of eigenvalues was first derived by Berkolaiko in [10] for discrete graphs with another proof given later by Colin de Verdière in [14]. A generalization of the nodal-magnetic connection for quantum graphs was then given by Berkolaiko and Weyand in [11]. In order to state the nodal magnetic connection, we first observe that due to gauge invariance the magnetic field can be parameterized by its magnetic fluxes through cycles of the graph $\vec{\alpha} \in (\mathbb{R}/2\pi\mathbb{Z})^{\beta}$ (again β is the first Betti number). The nodal-magnetic theorem says that under some restrictions, the n^{th} eigenvalue $\lambda_n(\vec{\alpha})$ as a function of the fluxes has a critical point at $\vec{\alpha} = 0$ whose Morse index is exactly the nodal surplus $\sigma_n := \phi_n - n$.

It can be shown (see [5]) that the bands of the dispersion relation manifold of periodic quantum graphs are given by the functions $\lambda_n(\vec{\alpha})$ with the magnetic fluxes replacing the quasi-momenta (also known as Bloch parameters). In this context the Morse index of λ_n at $\vec{\alpha} = 0$ is a stability index (the dimension of the space on which the particle has a negative mass) and the nodal surplus distribution is the stability index distribution among all energy levels. Berkolaiko and Weyand show in [11] that every λ_n has a critical point at every point of the form $\vec{\alpha} \in \{0, \pi\}^{\beta}$ which we denote as the *corner points* of the torus.

In a recent work in progress we showed that the stability index distribution at a given corner point is the same as in 0 and hence equal to the nodal surplus distribution. Using Morse theory one can estimate the distribution of stability indices over all possible critical points for all energy levels. The latter two arguments relates the nodal surplus distribution to critical points of the dispersion relation manifold, and in particular we can deduce by calculation that if the nodal surplus distribution is not binomial then there are infinitely many non-corner critical points. The critical points of the dispersion relation manifold are physically important as they determine the spectral gaps and the conductance of the system. In future work we plan to investigate the non-corner critical points of the dispersion relation manifolds of different graphs.

Neumann count and statistics. In an analogue to the nodal partition of manifolds given an eigenfunction, a *Neumann partition* was introduced independently both in [19, 18] and was further developed in [7, 6]. Given an eigenfunction of a two dimensional manifold and two critical points, a maximum and a minimum, a *Neumann domain* is defined as the intersection of the stable manifold of the minimum with the unstable manifold of the maximum (namely the union of all gradient flows between the two points). The Neumann partition is the partition of the manifold into such Neumann domains. The name *Neumann domain* is due to the fact that the eigenfunction restricted to a Neumann domain is an eigenfunction satisfying Neumann boundary conditions. In [3] we propose a similar Neumann partition of a quantum graph and a Neumann count $\mu_n := \#\{x : f_n'(x) = 0\}$ given by the number of critical points of the n^{th} eigenfunction. In analogue to the work on the nodal count, we proved explicit bounds on the Neumann count

$$n + c(\beta, \partial\Gamma) \le \mu_n \le n + C(\beta, \partial\Gamma),$$

which depend only on the first Betti number β and the boundary of the graph $\partial\Gamma$ which we define as the number of vertices of degree one. These bounds prove the asymptotic behavior $\mu_n \sim n$. In analog to the nodal surplus we define the Neumann surplus $\omega_n := \mu_n - n$ and show that the following limits exist

$$P(\omega_n = j) = \lim_{N\to\infty} \frac{\#\{n \le N : \omega_n = j\}}{N},$$

and define a symmetric probability. As a corollary, one can deduce both β and $|\partial\Gamma|$ from the nodal count and Neumann count. As there are only finitely many graphs with a given β and $|\partial\Gamma|$ this forms a great step forward towards the solution of the spectral inverse problem on graphs. We believe that isospectral graphs might be distinguished by their nodal and Neumann count together.

In a recent work in progress we have shown that 3 regular finite tree graphs have a binomial Neumann surplus distribution. This led us to **conjecture** both the existence of a Neumann analog to the nodal-magnetic connection and the convergence of the Neumann surplus distributions of sequences of graphs with $|\partial\Gamma| \to \infty$ to normal distribution.

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