RESEARCH STATEMENT

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My research subjects all lie under the illusive name *mathematical physics*. My main subject of research is spectral geometry of metric graphs, also known as *quantum graphs*. Another, surprisingly related, subject I work on is *Fourier Quasi-Crystals*. The relation between these two subjects goes through *stable polynomials*, as was recently showed by Sarnak and Kurasov [KS20]. My research includes questions and methods that relates to other mathematical disciplines such as

(1) Graph theory.
(2) Statistics and Probability in high dimensions.
(3) Riemannian Geometry.
(4) Stable Polynomials (real algebraic geometry).
(5) Dynamics and Ergodic theory.

The research statement is divided into sections according to subjects, each of which contains previous work, work in progress and future work.

1. Spectral geometry of metric graphs (quantum graphs)

A *metric graph* is a 1D manifold with singularities. The singular points are the vertices of a graph and every edge is a 1D segment. We denote such a metric graph by (Γ, ℓ), with graph structure given by Γ, a finite connected (discrete) graph[1] of $E > 1$ edges. The metric is determined by the vector of edge lengths $\ell = \{l_e\}_{e=1}^{E} \in \mathbb{R}_+^E$.

A *quantum graph* is a metric graph equipped with a 1D Schrödinger operator acting along the edges and a choice of vertex conditions such that the operator is self-adjoint. In the last few decades quantum graphs appeared in various scientific disciplines, modeling complex phenomena such as superconductivity in granular and artificial materials [Ale83], Anderson localization [CMV06, SS00], acoustic and electromagnetic waveguide networks [BK03], and quantum chaos [BGS02, KS97].

It is common to restrict the discussion to Laplace operator with Neumann-Kirchhoff vertex conditions, and we do so. The spectrum of (Γ, ℓ) is then the sequence of (square-root) eigenvalues of the positive Laplacian, denoted by

$$0 = k_0 < k_1 \le k_2 \le k_3 \ldots \nearrow \infty$$

including multiplicity. A pair (k_n, f_n) of (Γ, ℓ) is a pair of an eigenfunction f_n corresponding to the eigenvalue k_n. We often fix Γ and consider the change of such pairs as ℓ is changed.

This simple setting serves as non-trivial one-dimensional model for spectral geometry, where exotic mathematical phenomena can often occur. An example of such is the existence of scars (unusual localization of eigenfunctions) [BKW04, CdV15]. Another example, number theoretic in nature, is a recent work by Kurasov and Sarnak:

[1] For ease of presentation we restrict the discussion to simple graphs, i.e. every edge has a unique pair of two distinct vertices. We also assume no vertices of degree 2 as these are removable singularities.

Theorem 1.1. [KS] *Given a graph Γ there is a constant $C(\Gamma)$, such that for any $\mathbb{Q}$-independent ℓ, the spectrum of (Γ,ℓ) has infinite dimension over $\mathbb{Q}$ and does not contain any arithmetic progression of length $N > C(\Gamma)$.*

The term $\mathbb{Q}$-independent means that there is no (non-trivial) solution $q \in \mathbb{Q}^E$ to $\ell \cdot q = 0$.

1.1. **Generic properties of eigenvalues and eigenfunctions.** Let $\mathcal{P}$ denote some property of (k_n, f_n) pairs of (Γ,ℓ). A standard genericity statement has the form:

"*Given a graph Γ, for a generic choice of ℓ, every pair of (Γ,ℓ) satisfies $\mathcal{P}$*".

The seminal genericity result of Uhlenbeck, on compact Riemannian manifolds, serves as our motivational example:

Theorem 1.2. [Uhl72] *Given a compact manifold M, for a Baire-generic choice of smooth metric g, all (λ_n, f_n) pairs of (M, g) satisfy*

(1) λ_n is a simple eigenvalue (no multiplicity).
(2) 0 is not a critical value: $f_n(x) = 0 \Rightarrow \nabla f_n(x) \neq 0$.
(3) f_n is Morse: $\nabla f_n(x) = 0 \Rightarrow$ the Hessian of f_n at x is invertible.

Similar statements for metric graphs were proven by Friedlander and Berkolaiko-Liu

Theorem 1.3. [Fri05, BL17] *Given a graph Γ, for a Baire-generic ℓ, all (k_n, f_n) pairs of (Γ,ℓ) satisfy properties (1), (2) and (3) above. In addition,*

(4) f_n does not vanish on vertices.

Let $G \subset \mathbb{R}_+^E$ be the set of ℓ's satisfying all of the properties above. Then, Baire-genericity means that G contains a countable intersection of open-dense sets. Let us say *Strongly-generic* if the complement of G is contained in a countable union of lower dimensional varieties. One advantage of the stronger genericity is that G must have a full measure (unlike the Baire genericity). In a work in progress we show that

Theorem 1.4. [Alo] *Given a graph Γ, the set of ℓ's for which properties (1),(2),(3) and (4) hold for all pairs is **strongly-generic**.*

In [ABB18] we show that the implicit set of "good" ℓ's, can be replaced by an explicit criteria, if we replace *all pairs* with *almost every pair*, in the sense of a density 1 subsequence of pairs.

Theorem 1.5. [ABB18] *Given a graph Γ, for any $\mathbb{Q}$-independent ℓ, almost every pair of (Γ,ℓ) satisfies properties (1),(2),(3) and (4).*

One way of interpreting (4) is saying that generically eigenfunctions do not satisfy the "extra vertex condition" $f_n(v) = 0$. Vertex conditions are usually a system of $2E = \sum \deg(v)$ linear equations, $A \cdot \mathrm{trace}(f_n) = 0$, restricting the $4E$ dimensional vector $\mathrm{trace}(f_n)$ of all values and normal derivatives of f_n at the vertices. Intuitively one may expect that adding another linearly independent condition would result in an overdetermined system. Although this is not true, we can show that generically such intution is correct. Let us generalize the vertex conditions to a broader notion of homogeneous relations: $q(\mathrm{trace}(f_n)) = 0$ where q is a homogeneous polynomial. We say that a relation q is trivial (on Γ) if $q(\mathrm{trace}(f_n)) = 0$ for every pair of (Γ,ℓ), for any ℓ. In [Alo] we prove that

Theorem 1.6. [Alo] *Given a graph Γ and non-trivial homogeneous relation q,*

*(a) the set of ℓ's for which $q \neq 0$ for all (Γ,ℓ) pairs is **strongly-generic**.*
(b) for any $\mathbb{Q}$-independent ℓ, $q \neq 0$ for almost any pair of (Γ,ℓ).

Future work:

(1) Working on a conjecture of Sarnak, saying that generically the spectrum of (Γ, ℓ) should be $\mathbb{Q}$-independent.
(2) Working on a conjecture of Kottos and Smilansky that for $\mathbb{Q}$-independent ℓ, the spectrum should have a GOE behavior, when the graph size and complexity grows to infinity.
(3) Working on a conjecture of Quantum Unique Ergodicity for graphs with $\mathbb{Q}$-independent ℓ when the graph size and complexity grows to infinity.

1.2. **Nodal count and its universal statistics.** Given a manifold M, every eigenfunction f_n induce a *nodal partition* of M by removing the zero set $f_n^{-1}(0)$. The remaining connected components are called the *nodal domains* and the nodal count ν_n is the number of these nodal domains. The study of the nodal count is an integral part of spectral geometry. In 1923, Courant proved the upper bound $\nu_n \leq n$ [Cou23], and his student Stein showed that there cannot be any non-trivial lower bound by constructing a counter example [Ste25]. In 1956 Pleijel proved that Courant's bound cannot be obtained infinitely often, by proving an asymptotic bound $\limsup \frac{\nu_n}{n} \leq C < 1$ [Ple56]. These works led to many questions regarding the asymptotic behavior of the nodal count, most of which are still open. As a motivating example, we consider a conjecture made by Gnutzmann-Smilansky-Blum [BGS02] and Bogomolni-Schmit [BS02]

Conjecture 1.7. *Generically, for 2D manifolds and planar domains, ν_n is distributed like a Gaussian with mean and variance of order n.*

In the case of metric graphs, we conjecture that a similar universal behavior accrues in the limit of large graphs. Unlike the 2D case, we provide a proof for certain families of graphs and numerical evidence for other graph families.

The nodal count for a metric graph (Γ, ℓ) is defined in a similar manner. Generically, the Courant upper bound as shown in [GSW04]. Unlike the higher dimensional case, a generic lower bound was achieved in [Ber08], resulting in

$$n - \beta \leq \nu_n \leq n.$$

where $\beta = \operatorname{rank}(\pi_1(\Gamma))$ is the first Betti number of the graph (intuitively, the number of cycles). We denote the deviation from n by $\sigma(n) := n - \nu_n \in \{0, 1, \ldots, \beta\}$, and we ask how is it distributed as n grows. For a probabilistic interpretation of the nodal distribution question we view the error term $\sigma(n)$ of the nodal count as a random variable σ distributed over $\{0, 1, \ldots, \beta\}$. We prove in [ABB18] that for any (Γ, ℓ) with $\mathbb{Q}$-independent ℓ,

$$P(\sigma = j) := \lim_{N \to \infty} \frac{|\{n \leq N \, : \, \sigma(n) = j\}|}{N},$$

and that σ is symmetric with mean $\mathbb{E}(\sigma) = \frac{\beta}{2}$. From now we will always assume ℓ is $\mathbb{Q}$-independent. In general the nodal surplus distribution $\sigma = \sigma(\Gamma, \ell)$ depends on both Γ and ℓ. In a recent work [ABB21] we raise a conjecture of a universal behavior of the nodal surplus distribution as β grows to infinity.

Conjecture 1.8. [ABB21] *The distance between $\sigma(\Gamma, \ell)$ and the Gaussian distribution of same mean and variance goes to zero as $\beta \to \infty$, uniformly over all (Γ, ℓ) with Betti number β.*

A detailed and more quantified statement is given in [ABB21]. The first positive result is

Theorem 1.9. [ABB18] *If* Γ *has disjoint cycles (vertex disjoint) then* $\sigma(\Gamma,\ell)$ *is Binomial* $Bin(\beta,\frac{1}{2})$*. As a result, the conjecture holds for such graphs.*

The method of this theorem can be extended to graphs with disjoint clusters of cycles. In [ABB21] we prove the conjecture for two other families of graphs, using a very different approach. We also show that $\sigma(\Gamma,\ell)$ is convex in ℓ. We provide an upper bound, $C(\Gamma)$, to the distance of $\sigma(\Gamma,\ell)$ from the relevant Gaussian. This bound is uniform in ℓ, and can be numerically evaluated (efficiently). Using it we provide a vast numerical support for the conjecture, by showing how $C(\Gamma)$ decrease with β for a large number of graphs.

1.2.1. *Approaches to the conjecture.*

(1) Is there any relation between the nodal surplus distributions of a graph and its sub-graphs?
(2) Is there any model of random graphs for which $\sigma(\Gamma,\ell)$ can be estimated?
(3) In the proof of edge separated graphs having binomial distribution [ABB18], we prove that if a graph has a cut of one edge (an edge whose removal disconnects the graph) then this cut induces a partition of the random variable σ to a sum of two uncorrelated random variables. We believe that this proof can be extended to cuts of more edges, partitioning σ into uncorrelated random variables and a bounded error (that depends on the cut's size). This may provide a proof for Gaussian limiting distribution for graphs with many of small cuts.

1.3. **Neumann count.** In analogy to the nodal partition of manifolds given by an eigenfunction, a *Neumann partition* was introduced independently both in [Zel13, MF14] and was further developed in [BF16, BET17]. This partition, is a Morse partition of the manifold according to an eigenfunction. Connected components of the Neumann partition are called Neumann domains. The name is due to the fact that restricting an eigenfunctions to a Neumann domain gives a Laplace eigenfunction of that domain with Neumann boundary conditions. We define a metric graph analog in [ABBE20, AB21]. The n-th Neumann partition of a metric graph (Γ,ℓ) is given by removing the critical points along the edges. We denote the Neumann count by μ_n, namely the number of connected components of $\{x\in(\Gamma,\ell)\,:\,f_n'(x)\neq 0\}$. We prove in [AB21] upper and lower bounds on the Neumann count:

$$n+2-2\beta-|\partial\Gamma|\leq\mu_n\leq n+\beta,$$

where $|\partial\Gamma|$ is the number of degree 1 vertices of Γ. In analog to the nodal surplus we define the deviation from n, $\omega(n):=n-\mu_n$ and call it *Neumann surplus*[2]. Here too, we define the respective random variable ω, supported between $-\beta$ to $2\beta+|\partial\Gamma|-2$, and prove that for any (Γ,ℓ) with $\mathbb{Q}$-independent ℓ,

$$P\left(\omega=j\right):=\lim_{N\to\infty}\frac{|\{n\leq N\,:\,\omega(n)=j\}|}{N},$$

and that ω is symmetric around its mean $\mathbb{E}(\omega)=\frac{\beta+|\partial\Gamma|-2}{2}$.

Corollary 1.10. [AB21]

(1) The nodal and Neumann surplus distributions of (Γ,ℓ) *poses different information on* Γ*. For example, while all trees have the same nodal surplus sequence, trees with different* $|\partial\Gamma|$ *will have different* $\mathbb{E}(\omega)$*.*

[2]Although a priori it can be negative.

(2) *Having both $\mathbb{E}(\omega)$ and $\mathbb{E}(\sigma)$ gives β and $|\partial\Gamma|$, and thus bounding the number of edges and vertices of the graph.*

Similarly to the nodal distribution, we conjecture that the Neumann distribution has a universal Gaussian behaviuor as $\beta+|\partial\Gamma|$ grows to infinity. Following [ABB18] we prove a binomial result. A finite tree graph is d-regular if every vertex has degree either 1 or d.

Theorem 1.11. [AB21] *If Γ is a 3-regular finite tree and ℓ is $\mathbb{Q}$-independent, then $\omega(\Gamma,\ell)$ is Binomial, $Bin(|\partial\Gamma|-2,\frac{1}{2})$. Hence satisfies the Gaussian limit conjecture.*

1.3.1. *Future work.*

(1) Resolving isospectrality of quantum graphs. Are there two different graphs with the same nodal and Neumann count? In particular, can the nodal and Neumann count together distinguish between isospectral graphs?
(2) Is there a Neumann analog to the nodal-magnetic theorem. The result of 3 regular trees and its method of proof suggest that the analog of magnetic flux should relate to degree 3 vertices.
(3) As mentioned in [AB21], we believe that the bounds $-\beta\le\omega_n\le 2\beta+|\partial\Gamma|-2$ are not strict. We conjecture in [AB21] that the strict bounds are $0\le\omega_n\le\beta+|\partial\Gamma|-2$.

2. Quasi-crystals and Stable Polynomials

A crystal, from a physical point of view, is a system of atoms in a lattice structure, such as metals and semi-conductors for example. The lattice structure can be observed by a scattering experiment, the diffraction pattern would have picks at the dual lattice locations. Mathematically, this phenomena is described by the fact that a distribution of unit mass atoms along the lattice Λ, $\mu=\sum_{x\in\Lambda}\delta_x$, has a Fourier transform distribution $\widehat{\mu}=\sum_{k\in S}\delta_k$ which has unit mass atoms along the dual lattice S. This is known as the *Poisson summation formula*: for a rapidly decaying function f with Fourier transform $\hat{f}$,

$$\sum_{x\in\Lambda}f(x)=\sum_{k\in S}\hat{f}(k).$$

One may ask the *question:*

Is the above holds only for lattices?

What are the possible discrete sets Λ and S for which such relation holds?

Generalization of this form were considered by Meyer in the 70's but it was believed that no such physical system exists. A decade later, Shechtman discovered a *Quasi-crystal* in an experiment[3], a discovery for which he was awarded the 2011 Nobel prize.

In the context of metric graphs, a summation formula of similar nature was introduced by Kottos and Smilansky in [KS97]. It says that for any metric graph (Γ,ℓ) with set of periodic orbits $\mathcal{PO}$ and spectrum $\mathrm{spec}(\Gamma,\ell)$, for every "nice enough" function f,

$$\sum_{k\in\mathrm{spec}(\Gamma,\ell)}f(k)=\sum_{p\in\mathcal{PO}}c_{L_p}\hat{f}(L_p), \tag{2.1}$$

where L_p is the length of the periodic orbit p and c_{L_p} are complex coefficients. Kurasov and Sarnak had showed that (2.1) holds for all rapidly decaying functions and that the c_{L_p} coefficients grow slowly[4]. The measure $\mu=\sum_{k\in\mathrm{spec}(\Gamma,\ell)}\delta_k$ in such case is said to be a

[3]A scattering experiment where the diffraction pattern had sharp picks with a 5-fold symmetry which no lattice can have.

[4]That is $\sum_{p\in\mathcal{P}}|c_{L_p}|f(L_p)<\infty$ for any rapidly decaying f

Fourier Quasi-Crystal. This led Kurasov and Sarnak to the following observation about polynomials that do not vanish on the polydisc $\mathbb{D}^n$ and its open complement $(\mathbb{C} \setminus \overline{\mathbb{D}})^n$ which we call *stable polynomials*:

Theorem 2.1. [KS20] *Let $p \in \mathbb{C}[z_1, z_2 \dots z_n]$ be a stable polynomial, let $\ell \in \mathbb{R}_+^n$ and let*

$$\Lambda_{p,\ell} := \left\{ k \in \mathbb{R} \, : \, p(e^{ikl_1}, e^{ikl_2}, \dots, e^{ikl_n}) = 0 \right\},$$

counting including multiplicity. Then, the measure

$$\mu_{p,\ell} := \sum_{k \in \Lambda_{p,\ell}} \delta_k,$$

is a Fourier Quasi-Crystal.

The stability of p implies that $k \mapsto p(e^{ikl_1}, e^{ikl_2}, \dots, e^{ikl_n})$ is a *real rooted* trigonometric polynomial, that is $p(e^{ikl_1}, e^{ikl_2}, \dots, e^{ikl_n}) \neq 0$ when $\Im(k) \neq 0$. Olevskii and Ulanovskii showed that

Theorem 2.2. [OU20] *A measure $\mu := \sum_{k\in\Lambda} \delta_k$ is a Fourier Quasi-Crystal if and only if Λ is the zero set of real rooted trigonometric polynomial.*

In a work in progress, joint with Cynthia Vinzant, we conjecture that

Conjecture 2.3. [AV] *Given any real rooted trigonometric polynomial $F(k)$,*

*(1) There exists a **stable** polynomial p and a **positive** vector $\ell \in \mathbb{R}_+^n$ such that*

$$F(k) = p(e^{ikl_1}, e^{ikl_2}, \dots, e^{ikl_n}).$$

(2) If $F(k)$ is not periodic, then p and ℓ can be chosen such that ℓ is $\mathbb{Q}$-independent.

In [AV] we provide a "dictionary" between properties of stable polynomials and properties of Fourier quasi-crystals. We also show that some spectral properties of metric graphs can be translated to the Fourier quasi-crystals context. As an example, the Weyl law has the following analog:

Theorem 2.4. [AV] *Let p, ℓ and $\mu_{p,\ell}$ as in Theorem 2.1.Let $\boldsymbol{d} = (d_1, d_2 \dots, d_n)$ be the degrees of p in each variable. Then for any ball $B(x, R)$ around $x \in \mathbb{R}$ with radius $R > 0$,*

$$\mu_{p,\ell}(B(x,R)) = \frac{\boldsymbol{d} \cdot \ell}{\pi} R + \text{ error term},$$

with a uniform bound $|\boldsymbol{d}| = \sum d_j$ on the error term.

Another example is the analog of Theorem 1.1:

Theorem 2.5. [AV] *Let p, ℓ and $\Lambda_{p,\ell}$ as in Theorem 2.1 and let $\boldsymbol{d}$ be the degrees vector of p. Further assume that p is irreducible and has more then 2 monomials. Then $\Lambda_{p,\ell}$ has infinite dimension over $\mathbb{Q}$ and does not contain any arithmetic progression of length $N > C(\boldsymbol{d})$.*

The constant $C(\mathbf{d})$ is given explicitly and depends only on $\mathbf{d}$. More examples include

1. We prove the existence of a gap distribution in analogy to the work of Barra and Gaspard [BG00].
2. We prove a relation between the infimum of the gaps and the singularities of the zero variety of p in $\mathbb{C}^n$. In particular, if p is non singular, then $\Lambda_{p,\ell}$ is uniformly discrete[5].

[5] In fact a *Delone* set.

2.1. **Future work.**

(1) Working on Conjecture 2.3 using tropical geometry.
(2) We plan to analyze the gap distribution of $\mu_{p,\ell}$ as a function of p and ℓ, as an approach to proving the GOE conjecture for metric graphs.
(3) Constructing a model of random Fourier Quasi-Crystals in terms of random stable polynomials. An example can be $p(\vec{z}) = \det(1 - \mathrm{diag}(\vec{z})U)$ where U is a random unitary matrix, in which case we believe that the gap distribution will behaves like CUE.
(4) Higher dimensional analogs. Is it possible to create quasi-crystals in higher dimensions using stable varieties of higher co-dimension?

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